Machine Failures in Effective Process Time

Notation

X	=	natural process time (i.e., without failures), $E[X] = t_0$, $Var(X) = \sigma_0^2$
U_i	=	duration of i^{th} failure during X
N	=	number of failures during X
T	=	$X + \sum_{i=1}^{N} U_i$ = effective process time
m_f	=	mean time to failure (assumed exponential)
m_r	=	mean time to repair (general distribution), $E[U_i] = m_r$, $Var(U_i) = \sigma_r^2$
A	=	$m_f/(m_f + m_r)$ = availability
c_r	=	σ_r/m_r, repair time CV

Preliminaries

Since

$$\begin{aligned} Var(U_i) &= E[U_i^2] - E[U_i]^2 \\ \sigma_r^2 &= E[U_i^2] - m_r^2 \end{aligned}$$

we have

$$E[U_i^2] = \sigma_r^2 + m_r^2$$

Also, note that

$$E[U_iU_j] = E[E[U_iU_j]|U_i] = E[m_rU_j] = m_r^2, \quad i \neq j$$

And, since time to failure is assumed Poisson

$$\begin{aligned} E[N] &= X/m_f \\ Var(N) &= X/m_f \end{aligned}$$

Finally, we will need that

$$\begin{aligned} E[N(N-1)|X] &= E[N^2 - N|X] \\ &= Var(N|X) + E[N|X]^2 - E[N|X] \\ &= \frac{X}{m_f} + \frac{X^2}{m_f^2} - \frac{X}{m_f} \\ &= \frac{X^2}{m_f^2} \end{aligned}$$

Calculations

$$
\begin{aligned}
E[T] &= E_X[E_N[E[T|N,X]]] = E_X[E_N[E[X+\sum_{i=1}^{N} U_i]]|X] \\
&= E_X[E_N[X+Nm_r]|X] \\
&= E_X[X+\frac{X}{m_f}m_r] \\
&= t_0(1+\frac{m_r}{m_f}) \\
&= t_0(\frac{m_f+m_r}{m_f}) \\
&= \frac{t_0}{A}
\end{aligned}
$$

$$
\begin{aligned}
E[T^2] &= E_X[E_N[E[T^2|N,X]]] \\
&= E_X[E_N[E[(X+\sum_{i=1}^{N} U_i)^2]]|X] \\
&= E_X[E_N[E[X^2+\sum_{i=1}^{N} U_i^2+\sum_{i\neq j} U_iU_j+2\sum_{i=1}^{N} XU_i]]|X] \\
&= E_X[E_N[X^2+NE[U_i^2]+N(N-1)m_r^2+2NXm_r]|X] \\
&= E_X[E_N[[X^2+N(m_r^2+\sigma_r^2)+N(N-1)m_r^2+2NXm_r]|X] \\
&= E_X[X^2+\frac{X}{m_f}(m_r^2+\sigma_r^2)+\frac{X^2}{m_f^2}m_r^2+2\frac{X^2}{m_f^2}m_r] \\
&= E_X[X^2(1+\frac{m_r^2}{m_f^2}+2\frac{m_r}{m_f})+X(\frac{m_r^2}{m_f}+\frac{\sigma_r^2}{m_f})] \\
&= E_X[X^2(1+\frac{m_r}{m_f})^2+\frac{X}{m_f}(m_r^2+\sigma_r^2)] \\
&= \frac{E[X^2]}{A^2}+\frac{E[X]}{m_f}(m_r^2+\sigma_r^2)] \\
&= \frac{(\sigma_0^2+t_0^2)}{A^2}+\frac{t_0}{m_f}(m_r^2+\sigma_r^2)]
\end{aligned}
$$

$$
\begin{aligned}
Var(T) &= E[T^2] - E[T]^2 \\
&= \frac{\sigma_0^2 + t_0^2}{A^2} + \frac{t_0}{m_f}(m_r^2 + \sigma_r^2) - \frac{t_0^2}{A^2} \\
&= \frac{\sigma_0^2}{A^2} + \frac{t_0}{m_f}(m_r^2 + \sigma_r^2)
\end{aligned}
$$

$$
\begin{aligned}
c_e^2 &= \frac{Var(T)}{E[T]^2} \\
&= \frac{(\sigma_0^2/A^2) + (t_0/m_f)(m_r^2 + \sigma_r^2)}{t_0^2/A^2} \\
&= \frac{\sigma_0^2}{t_0^2} + \frac{A^2(m_r^2 + \sigma_r^2)}{m_f t_0} \\
&= c_0^2 + \frac{A^2\sigma_r^2}{m_f t_0} + \frac{A^2 m_r^2}{m_f t_0} \\
&= c_0^2 + \frac{A(m_f/(m_f + m_r))c_r^2 m_r^2}{m_f t_0} + (\frac{m_f}{m_f + m_r})(\frac{m_r}{m_f + m_r})\frac{m_r}{t_0} \\
&= c_0^2 + A(1-A)\frac{m_r}{t_0}c_r^2 + A(1-A)\frac{m_r}{t_0} \\
&= c_0^2 + (1 + c_r^2)A(1-A)\frac{m_r}{t_0}
\end{aligned}
$$

If repair times are exponential, then $c_r = 1$, so we get

$$
c_e^2 = c_0^2 + 2A(1-A)\frac{m_r}{t_0}
$$